My Problem Collection

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Contents

❇ Note :

- ❖ This document is NOT published from the Indian Statistical Institute. Rather, it is just a handout written by me on the basis on my personal experiences.
- ❖ All the coloured texts you'll see in the document are clickable links. By clicking on the text, the reader will be redirected to their corresponding websites.
- ❖ You can email me or send me a private message on AoPS for contacting me in case of an urgency, some serious problem or some typos in the paper.
- ❖ Every problem in this paper is collected from some known or unknown source. That source may be seniors, friends, books, past examinations or discussions in class. None of the problems are created by me. It's just a collection of collected problems.

❇ My Blogs :

- ❖ WordPress Blog : **A Mathematical Journey**
- ❖ AoPS Blog : **A Highly Random Blog**

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Happy Problem Solving & All the Best !

✻ Introduction

I am Jyotishka Ray Choudhury from Kolkata, West Bengal, India. I'm a student pursuing B.Stat (Hons.) at the Indian Statistical Institute (ISI), Kolkata. I reside at Birati, an outskirt of Kolkata. I did my schooling from Ramakrishna Mission, Rahara.

This motivation behind this problem collection is to serve as a source of problems for ISI aspirants aiming to crack the B.Stat or B.Math entrance exam. In spite of the fact the ISI is an *Institute of National Importance* and it offers unbelievable scope for both research works and high-paying jobs, it's little known among students. Naturally, a very little number of resources are available for preparation and scarcity of "ISI Entrance Level" problems is often faced by aspiring students. I hope this collection may help them overcome this problem.

✻ A Few Suggestions

At the undergraduate level, the institute offers 3-year B.Stat and B.Math programmes and the entrance exams of both the programmes are purely based on high school mathematics and logical reasoning. For details of the entrance exam, one may visit the admissions page of ISI Kolkata.

While trying the problems in this paper, you'll find some problems which are easy, whereas you'll find some problems to be tough. There's a common misconception that the problems asked in ISI are extremely tough. But, in reality, most of the problems are tricky and a few observations lead to the correct solution. Also, proof-writing is an important thing. So, there's no point in ignoring the easier problems. In fact, generally easier problems often have a more sophisticated solution. So, I'll suggest that in the entrance exam (In the subjective paper), it's better to complete less number of problems perfectly than to try all problems with incomplete solutions or bad representations.

As of now, you can try these problems. Later on, I intend to write handouts about preparation for ISI nad CMI entrance exams which will be uploaded to my blogs at WordPress and AoPS. Stay tuned!

✻ The Problem Collection

Let's get straight to the point. Till now, there are 71 problems in the collection because I didn't get much time or problems. But I promise to update the collection on a regular basis. Visit my website on WordPress for seeing the latest version. Happy Problem Solving !

1) Define a sequence $\{I_n\}$ such that

$$I_1 = \int_0^1 \frac{dx}{1+\sqrt{x}}$$

$$I_2 = \int_0^1 \frac{dx}{1+\frac{1}{1+\sqrt{x}}}$$

$$I_3 = \int_0^1 \frac{dx}{1+\frac{1}{1+\frac{1}{1+\sqrt{x}}}}$$

$$\vdots$$

Following this pattern, find the value of $\lim\limits_{n\to\infty} I_n$.

2) Find all integers k such that all roots of the polynomial

$$P(x) = x^3 - (k-3)x^2 - 11x + (4k-8)$$

are also integers.

3) Let a and m be arbitrary positive integers with $a \geqslant m$. Define a sequence $\{b_n\}$ such that

$$a^n \equiv b_n \pmod{m} \qquad \forall\ n \in \mathbb{N}$$

Prove that the sequence $\{b_n\}_{n\geqslant 1}$ is eventually periodic.

4) Let F_n be the n^{th} Fibonacci number, where $F_1 = F_2 = 1$. Show that $\exists\ k \in \mathbb{N}$ such that F_k ends with exactly 1893 zeroes.

[**Fact:** You can replace 1893 by any natural number of your choice and still, you'll see that that the result holds good.]

5) Let $a, b, c \in \mathbb{R}^+$ such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1$$

P.T. $abc \leqslant \frac{1}{8}$.

6) Draw 7 lines on the plane in an arbitrary manner. **P.T.** for any such configuration, 2 of the those 7 lines form an angle less than $26°$.

7) **P.T.** among any 7 real numbers, $\exists$ 2 numbers, say a and b such that

$$0 \leqslant \frac{a-b}{1+ab} \leqslant \frac{1}{\sqrt{3}}$$

8) A particle moves only towards above or towards right in each step. For $a, b, c, d, m \in \mathbb{Z}$, calculate the number of possible paths from the point (a, b) to (c, d) which do not touch or cut the line $y = m$, where $b < m < d$.

9) For $n \in \mathbb{N}$ and $n \geqslant 2$m, define T_n t be the number of non-empty subsets $\mathcal{S} \subseteq \{1, 2, \cdots, n\}$ such that the arithmetic mean of the elements of $\mathcal{S}$ is an integer. **P.T.** $(T_n - n)$ is even.

10) Let F_n be the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{n-k+1}{k} = F_{n+2}$$

11) For $n \in \mathbb{N} \cup \{0\}$, define $C_n := \dfrac{1}{n+1}\dbinom{2n}{n}$.

(a) Show that the number of ways to tile a stair-step shape of height n (Use your common sense to decode what this means) using exactly n rectangles is C_n.

(b) **P.T.**

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

12) Let $a, b, c, d, e, f \in \mathbb{N}$ such that

$$\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$$

Provided that $be = af + 1$, **P.T.** $d \geqslant b + f$.

13) For any given number of points on a plane, we define the *diameter* of the points as the maximum distance between any two points among the given points. Suppose, there are n points on a plane and the *diameter* of the points is λ.

P.T. we ca nalways a find a circle with radius $\frac{\sqrt{3}}{2}\lambda$ such that all the points lie within the circle.

14) Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function. Show that $\exists$ 3 natural numbers a, b, c in Arithmetic Progression such that $f(a) < f(b) < f(c)$ holds good.

15) Find all functions G satisfying the functional equation

$$G(x) + G\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$$

16) Solve the following equation in $\mathbb{R}$:

$$\left(\sqrt{2+\sqrt{2}}\,\right)^x + \left(\sqrt{2-\sqrt{2}}\,\right)^x = 2^x$$

17) Evaluate :

$$\int_{\pi}^{2\pi} \frac{(x^2+2)\cos x}{x^3} dx$$

18) Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable everywhere except possibly at $x = c$ (i.e. f may be or may not be diferentiable at $x = c$). Suppose $\lim_{x \to c} f'(x)$ exists and is equal to L.

P.T. f has a derivative at $x = c$ and $f'(c) = L$.

19) Let f and g be two functions such that

(a) $f, g : \mathbb{N} \to \mathbb{N}$,

(b) f is a surjective function,

(c) g is an injective funtion,

(d) $f(n) \geqslant g(n) \quad \forall\, n \in \mathbb{N}$.

P.T. $f \equiv g$. (You should know what that means)

20) Does there exist a surjective function $f : [a, b] \to [a, b)$ which is continuous on its domain ?

[**Fact:** In this kind of problem, if the answer is "yes", you'll have to show such an example. If the answer is "no", you need to **P.T.** the existence of such a function is impossible, and just an example won't suffice !]

21) **P.T.** $\exists\, m, n \in \mathbb{N}$ such that $(m + n + 1)$ is a perfect square and $(mn + 1)$ is a perfect cube.

22) Suppose, $\{a_n\}_{n\geqslant 1}$ is a sequence such that $a_1 \geqslant 1$ and

$$a_{k+1} \geqslant \frac{ka_k}{a_k^2 + k - 1} \qquad \forall\, k \geqslant 1$$

P.T. $\forall\, n \in \mathbb{N}, \ \displaystyle\sum_{i=1}^{n} a_i \geqslant n$.

23) $f : [0, a] \to \mathbb{R}$ is continuous, where $a \in \mathbb{R}^+$ and $\displaystyle\int_0^a f(x)\, dx = 0$.

P.T. $\exists\, c \in (0, a)$ such that $\displaystyle\int_0^c xf(x)\, dx = 0$.

24) $f : [0, 1] \to \mathbb{R}$ is continuous. Given, $f(0) = 0$ and $\displaystyle\int_0^1 f(x)\, dx = 0$,

P.T. $\exists\, c \in (0, 1)$ such that

$$c^2 f(c) = 2\int_0^c xf(x)\, dx$$

25) The real numbers $a_1, a_2, a_3, \cdots, a_{100}$ satisfy the following set of inequalities :

$$\begin{aligned} a_1 - 4a_2 + 3a_3 &\geqslant 0 \\ a_2 - 4a_3 + 3a_4 &\geqslant 0 \\ \vdots \qquad &\vdots \end{aligned}$$

$$\begin{aligned} a_{98} - 4a_{99} + 3a_{100} &\geqslant 0 \\ a_{99} - 4a_{100} + 3a_1 &\geqslant 0 \\ a_{100} - 4a_1 + 3a_2 &\geqslant 0 \end{aligned}$$

Provided that $a_1 = 1$, find the numbers $a_2, a_3, \cdots, a_{100}$ along with appropriate logical reasoning.

26) P is a polynomial with degree n and $\forall\ x \in \mathbb{R},\ f(x) \geqslant 0$. Define

$$H(x) := P(x) + P^{(1)}(x) + \cdots + P^{(n+1)}(x)$$

where $P^{(k)}(x)$ denotes the k^{th} derivative of P.

P.T. the function g is continuous and it doesn't have any real root.

27) Solve the following system in $\mathbb{R}$:

$$3\left(x + \frac{1}{x}\right) = 4\left(y + \frac{1}{y}\right) = 5\left(z + \frac{1}{z}\right)$$

where $xy + yz + zx = 1$.

28) $f : (0, \infty) \to \mathbb{R}$ satisfies the following :

(a) f is strictly increasing,

(b) $f(x) + \dfrac{1}{x} > 0 \qquad \forall\ x > 0$,

(c) $f(x) \cdot f\left(f(x) + \dfrac{1}{x}\right) = 1 \qquad \forall\ x > 0$

Determine all such functions f.

29) Find all continuous functions $f : (0, \infty) \to (0, \infty)$ such that $f(1) = 1$ and

$$\frac{1}{2}\int_0^x \big[f(t)\big]^2\, dt = \frac{1}{x}\left[\int_0^x f(t)\, dt\right]^2$$

30) If α and β are the roots of the the quadratic equation :

$$z + \frac{1}{z} = 2\big(\cos\theta + i\sin\theta\big) \qquad \forall\ \theta \in (0, \pi),$$

P.T. $\mid \alpha - i \mid = \mid \beta - i \mid$.

31) $x, y, z \in \mathbb{R}^+ \cup \{0\}$ such that $x + y + z = 1$. Find the maximum possible value of

$$x(x+y)^2(y+z)^3(z+x)^4$$

32) $f : \mathbb{R} \to (0, \infty)$ is a differentiable function which satisfies the equation $f'(x) = f(f(x))$ for all real numbers x. **P.T.** such a function can't exist.

33) **P.T.** $\exists$ a unique function $f : [0, \infty) \to (0, \infty)$ which satisfies the equation $f\big(f(x)\big) = 6x - f(x)$ for all real numbers x.

34) In $\triangle ABC$, the angular bisectors of $\angle A$, the median and the altitude through the vertex A, divide $\angle A$ into 4 equal parts. If $\angle B > \angle C$, then find the of measures of $\angle A, \angle B$ and $\angle C$.

35) Consider the sequence $\{a_n\}$ defined as

$$\{a_n\} = \Big\{n^2 + 50 \ : \ n \in \mathbb{N}\Big\}$$

If we take the GCD of each two consecutive terms of $\{a_n\}$, we obtain another sequence $\{b_n\}$, whose terms look like $3, 1, 1, 3, \cdots$

Find out the sum of all distinct elements of the sequence $\{b_n\}$.

36) $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(1) = 1$ and $f'(x) = e^{-xf(x)}$ holds for all $x \in \mathbb{R}$.

P.T. $\displaystyle\lim_{x\to\infty} f(x) < 1 + \frac{1}{e}$.

37) For any real number $\lambda \geqslant 1$, define $f(\lambda)$ to be the real solution to the equation $x(1 + \log x) = \lambda$.

P.T. $\displaystyle\lim_{\lambda\to\infty} \frac{f(\lambda)\log\lambda}{\lambda} = 1$.

38) **P.T.** if f is a continuous and differentiable function on $[a, b]$, then $\exists\ c \in (a, b)$ such that

$$\Big|f'(c)\Big| \ > \ \left|\frac{f(b) - f(a)}{b - a}\right|$$

provided that f is not a linear polynomial.

39) $P(x) \in \mathbb{Z}[x]$ is a monic polynomial. **P.T.** given any $n \in \mathbb{Z}$, $\exists\, M \in \mathbb{N}$ such that $P(n).P(n+1) = P(M)$.

40) Let I be the incentre of ΔABC. Let B' be the reflection of B w.r.t. the line AI. **P.T.** the circumcircle of $\Delta BCB'$ lies on the circumcircle of ABC.

41) $\square ABCD$ is a parallelogram such that e is the midpoint of AD and F is a point on CE such that $BF \perp CE$. **P.T.** ΔAFB is isosceles.

42) $f : \mathbb{R} \to \mathbb{R}$ is a continuous and decreasing function. **P.T.** f has a unique fixed point. Also, show that the set $A = \{x \in \mathbb{R} \,:\, f(f(x)) = x\}$ is either an infinite set or has odd number of elements.

43) $a, b, c \in \mathbb{R} \setminus \{0\}$ satisfy

$$a^2 + b + c = \frac{1}{a}$$

$$b^2 + c + a = \frac{1}{b}$$

$$c^2 + a + b = \frac{1}{c}$$

P.T. at least two of a,b,c must be equal.

44) $f : [a, b] \to \mathbb{R}$ is a differentiable function such that $f'(x)$ is continuous $\forall\, x \in (a, b)$ and $f(a) = 0$.

P.T. $\displaystyle\int_a^b \left[f(x)\right]^2 dx \;\leqslant\; (b-a)^2 \int_a^b \left[f'(x)\right]^2 dx$

45) For a given set $S \subset \mathbb{R}^2$, it is given that $|S| < \infty$. Given that :

(a) No three points in S are collinear.

(b) $A, B, C \subset S \iff A', B', C' \subset S$.

where A', B', C' are points diametrically opposite to A, B, C respectively with respect to the circumcircle of ΔABC.

P.T. all the points in S are concyclic.

46) $\forall\, x \in \mathbb{R}$, **P.T.** $\lfloor x \rfloor + \left\lfloor x + \dfrac{1}{2} \right\rfloor = \lfloor 2x \rfloor$.

47) **P.T.** there does not exist $P(x) \in \mathbb{Z}[x]$ such that $\deg(P) \geqslant 2$ and $P(x) \in \mathbb{R} \setminus \mathbb{Z}$ for all $x \in \mathbb{R}$.

48) **P.T.** $\forall$ acute angles A, B,

$$2 \leqslant \frac{\sin A}{\sin B} + \frac{\sin B}{\sin A} \leqslant \frac{A}{B} + \frac{B}{A} \leqslant \frac{\tan A}{\tan B} + \frac{\tan B}{\tan A}$$

holds good.

49) Find all functions f such that

$$f(x) \leqslant x \quad \text{and} \quad f(y+z) \leqslant f(y) + f(z)$$

holds $\forall\ x, y, z \in \mathbb{R}$.

50) For any given k points in a plane, define the "Maximal Length" of those points to be the maximum distance between any two points among the given points. Suppose, n points are in a plane with Maximal Length d.

P.T. we can always find a circle with diameter $\sqrt{3}d$ such that all of those n points lie inside the circle.

51) For any $a_1, a_2, \cdots, a_n \in \mathbb{R}$, **P.T.**

$$\frac{|a_1 + a_2 + \cdots + a_n|}{1 + |a_1 + a_2 + \cdots + a_n|} \leqslant \frac{|a_1|}{1 + |a_1|} + \frac{|a_2|}{1 + |a_2|} + \cdots + \frac{|a_n|}{1 + |a_n|}$$

52) **P.T.** $\exists$ infinitely many $n \in \mathbb{N}$ such that each of $n, (n+1), (n+2)$ can be written as a sum of 2 perfect squares.

53) f is a function satisfying

$$|f(x) - f(y)| \leqslant \frac{1}{2}\,|x - y| \qquad x, y \in \mathbb{R}$$

P.T. $\exists$ exactly one $\lambda \in \mathbb{R}$ such that $f(\lambda) = \lambda$.

54) $\{a_n\}$ is a sequence of real numbers such that $a_1 \geqslant 1$ and

$$a_{k+1} \geqslant \frac{k a_k}{a_k^2 + k - 1} \qquad \forall\ k \in \mathbb{N}$$

P.T. for all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} a_i \geqslant n$

55) $P(x) \in \mathbb{R}[x]$ satisfying

$$P(x^2) = P(x).P(x-1) \qquad \forall\ x \in \mathbb{C}$$

Find all such polynomials P.

56) $f : [0, \infty) \to [0, \infty)$ satisfies

$$x = f(x) \cdot e^{f(x)} \qquad \forall\ x \geqslant 0$$

Find the value of $\int_0^e f(x)\ dx$.

57) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with period 1.

(1) **P.T.** f is bounded and it achieves its bound somewhere.

(2) **P.T.** $\exists\ x_0 \in \mathbb{R}$ such that $f(x_0) = f(x_0 + \pi)$.

58) $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(1) = M$ and

$$f'(x) = \frac{1}{x^2 + [f(x)]^2} \qquad \forall\ x \in \mathbb{R}$$

P.T. $f(2020) < M + 1$.

59) Find all continuous $f : [0, 1] \to \mathbb{R}$ satisfying

$$\int_0^1 f(x)\ dx = \frac{1}{3} + \int_0^1 \big[f(x)\big]^2 dx$$

60) f is a function which is continuous, bounded on $[0, 1]$ and differentiable on $(0, 1)$. Also, $|f'(x)| \leqslant |f(x)|\ \forall\ x \in (0, 1)$.

P.T. $f(x) = 0 \quad \forall\ x$.

61) **P.T.** $\displaystyle\lim_{n\to\infty} n^2 \int_0^{\frac{1}{n}} x^{x+1}\ dx = \frac{1}{2}$

62) $f; \mathbb{R} \to \mathbb{R}$ is a continuous function. Define $g(x) = f(x) \int_0^x f(t)\ dt$. Prove that if g is a non-increasing function, then f is identically zero.

63) $f : \mathbb{R} \to \mathbb{R}$ is a continuous, odd function. Also, it is periodic with period T. Prove that for any $\lambda \in \mathbb{R}$, $\int_a^x f(t)\ dt$ is also periodic and find its period.

64) $f : [a, b] \to \mathbb{R}$ is a differentiable function such that $f(a) = 0$.

P.T. $\displaystyle\int_a^b \left[f(x)\right]^2 dx \leqslant \frac{(b-a)^2}{2} \int_a^b \left[f'(x)\right]^2 dx$

65) $P(x) \in \mathbb{Z}[x]$ and $\deg(P) = n$. All the roots $x_1, x_2, \cdots, x_n \in \mathbb{R}$ and all of them are positive.

Prove that all the roots belong to the set

$$R = \left\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, 1, 2\right\}$$

66) Let f be a function $f(xy) = f(x)f(y)\ \forall\ x \in \mathbb{R}$ and $f(1+x) = 1 + x\big(1 + g(x)\big)$, where $\lim\limits_{x\to 0} g(x) = 0$ then find the value of

$$\int_1^2 \frac{f(x)}{f'(x)(1+x^2)}\, dx$$

67) Let x, y be real numbers such that the numbers $(x+y), (x^2+y^2)$ and (x^3+y^3) are integers. Prove that for all natural n, the number (x^n+y^n) is an integer.

68) P is a real polynomial such that if α is irrational then $P(\alpha)$ is irrational. Prove that $\deg P \leqslant 1$.

69) In a $1 \times 6n$ grid, $4n$ squares are coloured red and $2n$ squares are colored blue. Prove that there exists $3n$ consecutive squares in the grid that contains $2n$ red squares and n blue squares.

70) Find all continuous functions $f : \mathbb{R} \mapsto \mathbb{R}$ satisfying

$$f(x+y+a) + b = f(x) + f(y) \quad \forall\, x, y \in \mathbb{R}$$

where a and b are fixed real numbers.

71) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) - f(y) \ \in\ \mathbb{Q} \qquad \forall\ \ (x-y)\ \in\ \mathbb{Q}$$

✻ Some Helpful Resources

There are a handful of resources available for your preparation. Here, I have tried to include a number of books and some other online resources which, I guess, may help you in your preparation.

✔ **Online Resources :**

a) **The Art of Problem Solving (AoPS) Community**

b) **Prof. K.D. Joshi's Blog at Blogspot**

c) **Aditya Ghosh's Blog at WordPress**

d) **Evan Chen's Webpage**

✔ **Books :**

a) **Test of Mathematics at the 10+2 Level** (ISI)

b) **Art & Craft of Problem Solving** *by* Paul Zeitz

c) **Putnam & Beyond** *by* Titu Andreescu

d) **An Excursion in Mathematics** (Bhaskaracharya Pratishthana)

e) **Problem Solving Strategies** *by* Arthur Engel

f) **Introduction to Real Analysis** *by* Bartle & Sherbert